Linear Algebra

UNIT-1

MATRICES & GAUSSIAN ELIMINATION

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LINEAR ALGEBRA

Linear Equations

- equation in n variables in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $x_1, x_2 \dots x_n$ are unknown variables,
$a_1, a_2 \dots a_n$ are coefficients and
b is a constant

System of Linear Equations

SYSTEM of LINEAR EQUATIONS

- INCONSISTENT
  no solution exists
- CONSISTENT
  - one unique solution
  - infinitely many solutions

- set of m equations and n variables

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}$$

MATRIX REPRESENTATION

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

A — coefficient matrix

X — matrix of unknowns

b — matrix of right-side constants

- a_{ij} : component of i^{th} row and j^{th} column
- if $m = n$, square matrix of n equations and n unknowns
- if all b's are zero, homogeneous system of equations;
  if any one b is nonzero, non-homogeneous system of equations

AUGMENTED MATRIX

$$\text{Matrix } [A:b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : b_2 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : b_n \end{bmatrix}_{m \times (n+1)}$$

SINGULAR MATRIX

Matrix whose determinant is 0

Solution to System of Linear Equations

ELEMENTARY ROW TRANSFORMATIONS

1) $R_i \longrightarrow kR_i \quad (k \neq 0)$
   multiply entries of a row by non-zero scalar

2) $R_i \longrightarrow R_i + kR_j \quad (k \neq 0)$
   sum of itself and non-zero scalar multiple of another row

3) $R_i \longleftrightarrow R_j$
   swap two rows

$$AX = BI$$

to transform

$$IX = BA^{-1}$$

EQUIVALENT MATRICES

- if two matrices A and B are such that each of them can be obtained from the other by a definite number of elementary transformations, they are said to be equivalent

- $A \ \square \ B$

Echelon Form of a Matrix

- A rectangular matrix $A_{m \times n}$ is said to be in echelon form if it satisfies the following conditions

1. First non-zero element of each row is called pivot element

2. All entries below the pivot in its column must be 0

3. Each pivot lies right to the pivot of the previous row (produces staircase pattern)

4. zero rows (if they exist) lie at the bottom of the matrix

- Example:

$$\begin{bmatrix} a & b & c & d \\ 0 & 0 & e & f \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

- if square matrix, Upper Triangular Matrix : determinant $= P_1 \times P_2 \cdots P_n$

ROW REDUCED ECHELON FORM (RREF)

- Every row in echelon form must be divided by its pivot such that the first nonzero element is always 1

convert to 0

denoted by R

$$\begin{bmatrix} 1 & b/a & c/a & d/a \\ 0 & 0 & 1 & f/e \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4} = R$$

all remaining elements in pivot columns are 0

geometry of LE

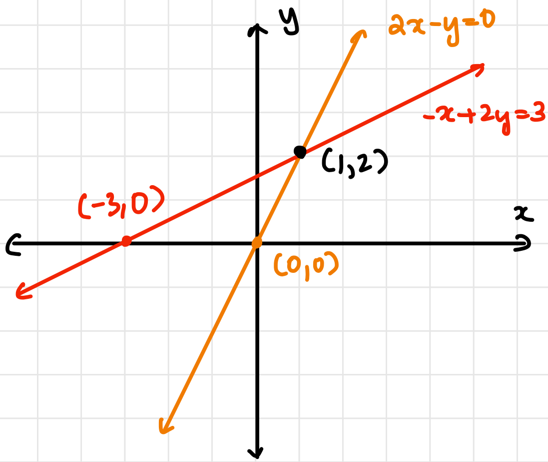
Row Picture

- 2 variables and 2 equations
- 2 straight lines in two dimensions
- solution: unique point of intersection of lines

Q1. Solve, show row picture

$$2x - y = 0$$
$$-x + 2y = 3$$

row picture



$$\underset{A}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

$2x - y = 0$

x	0	1
y	0	2

$-x + 2y = 3$

x	-3	1
y	0	2

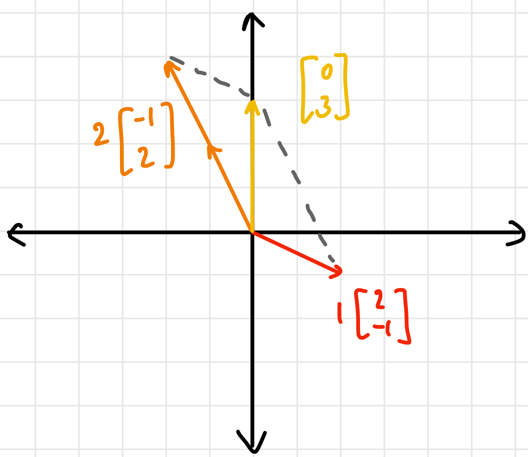
Column Picture

- combination of column vectors on the left side that produces right hand side

Q2. Solve, show column picture

$$2x - y = 0$$
$$-x + 2y = 3$$

column picture



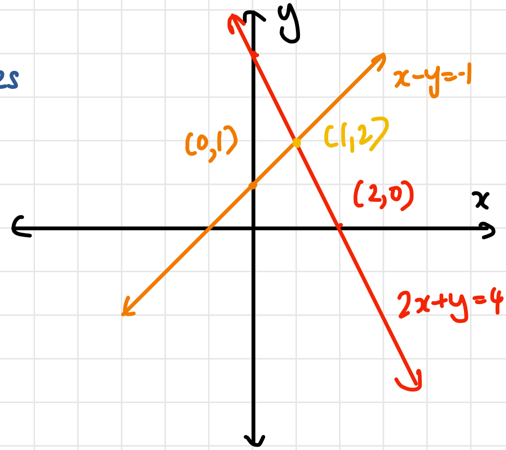
linear combination of columns

$$x\underset{\text{column 1}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} + y\underset{\text{column 2}}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

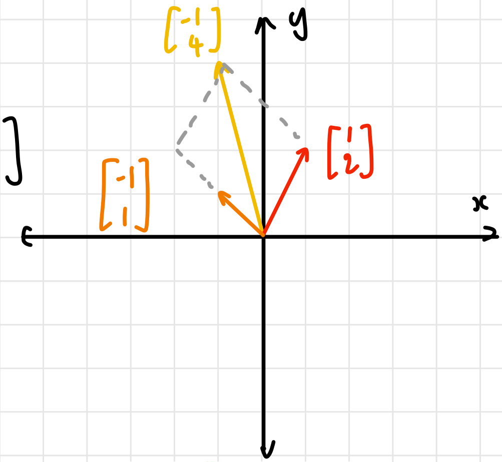
$$1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Q3 Solve and show row & column pictures

$$x - y = -1$$
$$2x + y = 4$$



Row picture: solving, we get (1,2)
unique solutions
two lines intersect at (1,2)

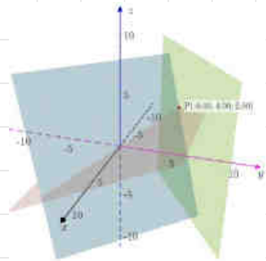


Column picture: linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ gives $\begin{bmatrix}-1\\4\end{bmatrix}$

Three Dimensions

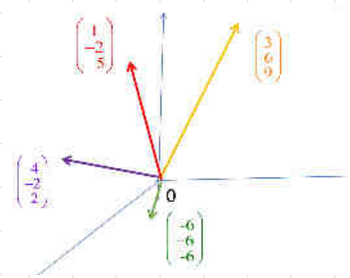
Row Picture

intersection of 3 planes at a point (unique solution)



Column Picture

linear combination of vectors to form parallelopiped

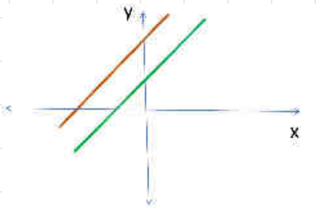


Equation of Line

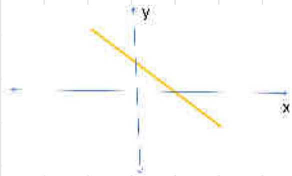
A line in n dimensions requires n-1 equations $(n \geq 2)$

Singular cases in 2-D

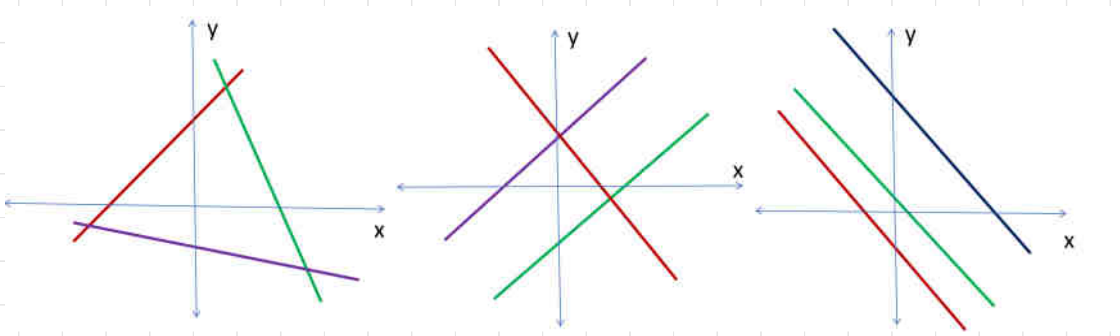
1. Two lines are parallel (no solution)



2. Two lines are coincident (same line, infinite solutions)

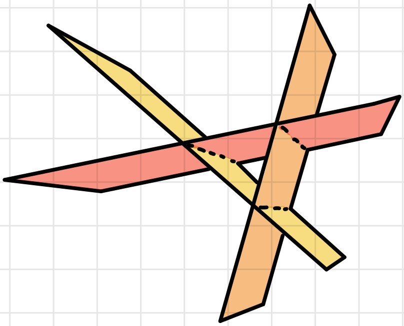


Three lines

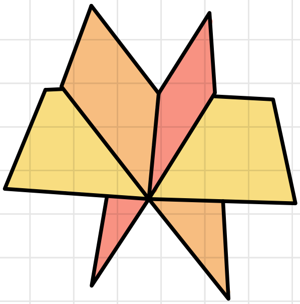


Singular cases in 3-D

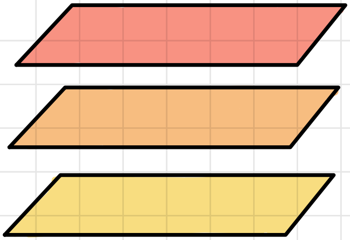
1. Every pair of lines interset in a line and all those lines are parallel



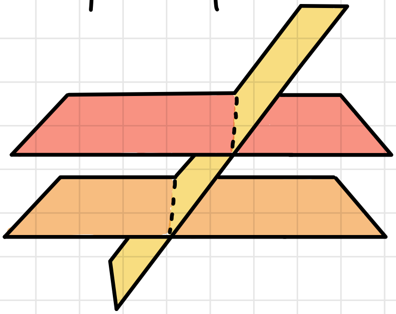
2. Three planes have a line in common



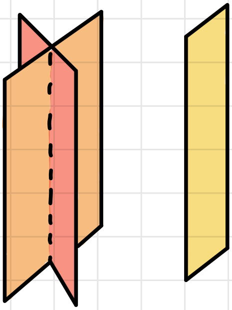
3. All three planes parallel



4. Two planes parallel



5. Two planes intersect in a line and third parallel



6. All three planes overlap

Column Picture

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \Rightarrow x_1\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + x_3\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- each column vector is position vector with origin as a point
- these col vectors lie on a plane (pass through origin)
- every combination of these vectors on LHS lie in the same plane (3 vectors coplanar)
- if vector b is not on the plane, system is singular and has no solution
- if vector b is on the plane, infinite number of solutions

GAUSSIAN ELIMINATION

— rank of matrix

- a square matrix A of order n is said to have rank r if
  - at least one minor of order r does not vanish (sub-determinant not 0)
  - every minor of order r+1 vanishes
- rank of matrix A is denoted by rank(A) = r
- no. of nonzero rows in echelon form of A

Q4. Find the rank of the following

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$ $r=2$ (b) $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $r=1$ (c) $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $r=0$

(d) $D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ $r=2$ (e) $E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $r=3$ (f) $F = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $r=1$

(g) $G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $r=2$

Relationship Between Rank, Consistency and Solution

if $\text{rank}(A) = r$, then

1. if $\text{rank}(A) = \text{rank}(A:b) = r$, system $AX = b$ is consistent and has a solution

2. if $\text{rank}(A) = \text{rank}(A:b) = r = n$, system $AX = b$ is consistent and has a unique solution

3. if $\text{rank}(A) = \text{rank}(A:b) = r < n$, system $AX = b$ is consistent and has infinite no. of solutions

4. if $\text{rank}(A) \neq \text{rank}(A:b)$, system $AX = b$ is inconsistent and has no solution

Gaussian Elimination

- check for consistency and solve linear equations

- for given system of LE $AX = b$ apply elementary row transformations to the augmented matrix $[A:b]$ and reduce it to $[U:c]$ where U is an Upper Triangular matrix

- We get an equivalent system $UX = c$ which can be solved by backward substitution

- Here A and U are Equivalent matrices and hence solution of $AX = b$ is the same as solution of $UX = c$

Steps for Elementary Row Transformations in GE

1. No exchange of rows

2. First row should be unaltered

3. First nonzero element in nonzero row is called pivot

4. $Ax=b$ and $Ux=c$ have same solution

example

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

$$[A:b] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix} \xrightarrow[R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1]{R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1} \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & d_{22} & d_{23} & : & c_2 \\ 0 & d_{32} & d_{33} & : & c_3 \end{bmatrix}$$

$$\xrightarrow{R_3 - \left(\frac{d_{32}}{d_{22}}\right)R_2}$$

$$[U:c] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & d_{22} & d_{23} & : & c_2 \\ 0 & 0 & e_{33} & : & c_4 \end{bmatrix}$$

Qs: check for consistency and solve if consistent

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$
$$2x_1 + 2x_2 - 3x_3 + x_4 = 3$$
$$3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$$

$$[A:b] = \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$n = 4$

$r([A:b]) = 2$

$r(A) = 2$

$$r(A) = r([A:b]) < n$$

$\therefore$ consistent with infinite no. of solutions

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$
$$x_3 - 7x_4 = -7$$

Q6. check for consistency and solve if consistent

$$x_1 + x_2 + x_3 + x_4 = 4$$
$$2x_1 + 3x_2 + 3x_3 - x_4 = 3$$
$$5x_1 + 7x_2 + 4x_3 + x_4 = 5$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & -1 & -4 & -15 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & -3 & 2 & -5 \end{array}\right]$$

$n = 4 \qquad r(A) = 3 \qquad r([A:b]) = 3$

$\therefore$ consistent with infinite no. of solutions

$+3 - \frac{2}{3}$

$$x_1 + x_2 + x_3 + x_4 = 4$$
$$x_2 + x_3 - 3x_4 = -5$$
$$-3x_3 + 2x_4 = -5$$

$\frac{-15}{3} \; \frac{-25}{3}$

Let $\boxed{x_4 = k}$

$$-3x_3 + 2k = -25$$
$$-3x_3 = -2k - 25$$

$$\boxed{x_3 = \frac{2}{3}k + \frac{25}{3}}$$

$$x_2 + \frac{2}{3}k + \frac{25}{3} - 3k = -5$$

$$\boxed{x_2 = \frac{-40k}{3} + \frac{7}{3}}$$

$$x_1 - \frac{40k}{3} + \frac{7}{3} + \frac{2}{3}k + \frac{25}{3} + k = 4$$

$$\boxed{x_1 = \frac{-20}{3} + \frac{35}{3}}$$

Q7. check for consistency and solve if consistent

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 5x_2 - x_3 = -4$$
$$3x_1 - 2x_2 - x_3 = 5$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 + 8R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

$n = 3$

$r(A) = 3$

$r([A:b]) = 3$

$\therefore$ consistent with unique solution

$$x_1 + 2x_2 + x_3 = 3$$
$$x_2 - 3x_3 = -10$$
$$-28x_3 = -84$$

$\boxed{x_3 = 3}$

$x_2 - 9 = -10$

$\boxed{x_2 = -1}$

$x_1 - 2 + 3 = 3$

$\boxed{x_1 = 2}$

Q8. check for consistency and solve if consistent

$$2x - 3y + 2z = 1$$
$$5x - 8y + 7z = 1$$
$$y - 4z = 3$$

$$[A:b] = \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \\ 0 & 1 & -4 & 3 \end{array}\right] \xrightarrow{R_2 - 5/2 R_1} \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 1 & -4 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r(A) = 2$$
$$r(A:b) = 2$$
$$n = 3$$

$\therefore$ consistent with infinite no. of solutions

$$2x - 3y + 2z = 1$$
$$-\frac{1}{2}y + 2z = -3/2$$

Let $z = k$

$$-\frac{y}{2} + 2k = -3/2$$
$$-\frac{y}{2} = -\frac{3}{2} - 2k$$
$$y = 3 + 4k$$

$$2x - 3(3+4k) + 2k = 1$$
$$2x - 9 - 12k + 2k = 1$$
$$2x = 10 + 10k$$
$$x = 5k + 5$$

$$(x, y, z) = (5k+5,\ 3+4k,\ k)$$

Q9. check for consistency and solve if consistent

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 5x_2 - x_3 = -4$$
$$3x_1 - 2x_2 - x_3 = 5$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 + 8R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

$r(A) = 3$

$r(A:b) = 3$

$n = 3$

$\therefore$ consistent with unique solution

$$x_1 + 2x_2 + x_3 = 3$$
$$x_2 - 3x_3 = -10$$
$$-28x_3 = -84$$

$-28x_3 = -84$

$\boxed{x_3 = 3}$

$x_2 - 9 = -10$

$\boxed{x_2 = -1}$

$x_1 - 2 + 3 = 3$

$\boxed{x_1 = 2}$

Q10. check for consistency and solve if consistent

$$x_1 + x_2 - 2x_3 + 3x_4 = 4$$
$$2x_1 + 3x_2 + 3x_3 - x_4 = 3$$
$$5x_1 + 7x_2 + 4x_3 + x_4 = 5$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

$n = 4$
$r(A) = 2$
$r([A:b]) = 3$

$\therefore$ inconsistent and no solution

Q11. check for consistency and solve if consistent

$$x + y - z = 2$$
$$x + 2y + z = 3$$
$$x + y + (a^2 - 5)z = a$$

$$\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

if $a \neq \pm 2$

$n = 3$
$r(A) = 3$
$r(A:b) = 3$

$\therefore$ consistent with unique solution

$$x + y - z = 2$$
$$y + 2z = 1$$
$$(a^2 - 4)z = a - 2$$

$$z = \frac{a-2}{a^2-4} = \frac{1}{a+2}$$

$$y + \frac{2}{a+2} = 1$$

$$y = \frac{a+2-2}{a+2} = \frac{a}{a+2}$$

$$x + \frac{a}{a+2} - \frac{1}{a+2} = 2$$

$$x = \frac{2a+4-a+1}{a+2} = \frac{a+5}{a+2}$$

$$(x, y, z) = \left(\frac{a+5}{a+2}, \frac{a}{a+2}, \frac{1}{a+2} \right)$$

If $a = 2$ $\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$r(A) = 2 = r(A:b) < n$$

$\therefore$ consistent with infinite no. of solutions

$$x + y - z = 2$$
$$y + 2z = 1$$

Let $z = k$

$$y + 2k = 1$$
$$y = 1 - 2k$$

$$x + 1 - 2k - k = 2$$
$$x + 1 - 3k = 2$$
$$x = 1 + 3k$$

$$(x, y, z) = (1+3k, 1-2k, k)$$

If $a = -2$ $\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$

$r(A) = 2$
$r(A:b) = 3$ $\quad\therefore$ inconsistent and no solution

Q12. $x + z = 1$
$x + y + z = 2$
$x - y + z = 1$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2}$$

$$[U:C] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$r(A) = 2 \quad r([A:b]) = 3 \quad n = 3$

$\therefore$ inconsistent and no solution

Q13. check for consistency and solve if consistent

$$x+y+z=8$$
$$2x-3y+4z=3$$
$$3x-y-3z=6$$

$$[A:b]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 2 & -3 & 4 & 3 \\ 3 & -1 & -3 & 6 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & -5 & 2 & -13 \\ 0 & -4 & -6 & -18 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{4}{5}R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & -5 & 2 & -13 \\ 0 & 0 & -\frac{38}{5} & -\frac{38}{5} \end{array}\right]$$

$$r(A)=3=r([A:b])=n$$

$\therefore$ consistent, unique solution

$$-\frac{38}{5}z=-\frac{38}{5}$$
$$\boxed{z=1}$$

$$-5y+2=-13$$
$$-5y=-15$$
$$\boxed{y=3}$$

$$x+3+1=8$$
$$\boxed{x=4}$$

$$(x,y,z)=(4,3,1)$$

Breakdown of Elimination

- if a zero appears in pivot position, elimination needs to stop temporarily or permanently
- if problem can be cured & elimination can proceed, system is non-singular
- if breakdown is unavoidable/permanent, system is singular and has no solution/infinitely many solutions
- Non-singular and curable ($|A| \neq 0$)
- Singular and incurable ($|A| = 0$)
- Singular ($|A| = 0$)

Q14. check for consistency and solve if consistent

$x+y+z=-3$
$2x+2y+5z=6$
$4x+6y+8z=7$

pivot is 0

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 2 & 2 & 5 & 6 \\ 4 & 6 & 8 & 7 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & 0 & 3 & 12 \\ 0 & 2 & 4 & 19 \end{array}\right]$$

swap rows

$R_2 \leftrightarrow R_3$

non-singular & curable

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & 2 & 4 & 19 \\ 0 & 0 & 3 & 12 \end{array}\right]$$

$3z = 12$ $\quad$ $\boxed{z = 4}$

$2y + 16 = 19$ $\quad$ $\boxed{y = \frac{3}{2}}$

$x + \frac{3}{2} + 4 = -3$ $\quad$ $\boxed{x = \frac{-17}{2}}$

$$(x, y, z) = \left(\frac{-17}{2}, \frac{3}{2}, 4\right)$$

Qus. check for consistency and solve if consistent

$$x + y + z = 6$$
$$x + y + 3z = 10$$
$$x + 2y + 4z = 12$$

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 2 & 4 & 12 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

non-singular & curable

$2z = 4$ $\quad$ $\boxed{z = 2}$

$y + 3 \times 2 = 6$ $\quad$ $\boxed{y = 0}$

$x + 2 = 6$ $\quad$ $\boxed{x = 4}$

$$(x, y, z) = (4, 0, 2)$$

Q16. check for consistency and solve if consistent

$$x+y+z=6$$
$$x+y+3z=10$$
$$x+y+4z=13$$

pivot = 0

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 1 & 4 & 13 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 13 \end{array}\right]$$

$$x+y+z=6$$
$$\left.\begin{array}{r} 2z=4 \\ 3z=13 \end{array}\right] \text{ impossible } (z=2 \text{ \& } z=13/2)$$

incurable and singular

Q17. check for consistency and solve if consistent

$$x+y+z=6$$
$$x+y+3z=10$$
$$x+y+4z=12$$

rows consistent

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 1 & 4 & 12 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{3}{2}R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

singular, curable

$$r(A) = 2 = r(A:b) < n$$

$\therefore$ consistent, infinite no. of solutions

$2z = 4$
$z = 2$

Let $y = k$

$x + k + 2 = 6$
$x = 4 - k$

$$(x, y, z) = (4-k, k, 2)$$

Q18. $u + v + w = -2$
$3u + 3v - w = 6$
$u - v + w = -1$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -2 \\ 3 & 3 & -1 & 6 \\ 1 & -1 & 1 & -1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & -2 \\ 0 & 0 & -4 & 12 \\ 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc:c} 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -4 & 12 \end{array}\right]$$

$-4w = 12$
$w = -3$

$-2v = 1$
$v = -1/2$

$u - 1/2 - 3 = -2$
$u = \frac{3}{2}$

$$(u, v, w) = (3/2, -1/2, -3)$$

Q19. For which 3 no.s 'a' will elimination fail?

$$ax + 2y + 3z = b_1$$
$$ax + ay + 4z = b_2$$
$$ax + ay + az = b_3$$

$$[A:b] = \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ a & a & 4 & b_2 \\ a & a & a & b_3 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & a-2 & a-3 & b_3 - b_1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & 0 & a-4 & b_3 - b_2 \end{array}\right]$$

$$a = 0, \quad a = 2, \quad a = 4$$

Q20. For what values of a and b does the following system have

(i) A unique solution

(ii) Infinitely many solutions

(iii) No solution

$$x + 2y + 3z = 2$$
$$-x - 2y + az = -2$$
$$2x + by + 6z = 5$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ -1 & -2 & a & -2 \\ 2 & b & 6 & 5 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & 0 & a+3 & 0 \\ 0 & b-4 & 0 & 1 \end{array}\right]$$

elimination fails

$$[U:c] \quad \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & b-4 & 0 & 1 \\ 0 & 0 & a+3 & 0 \end{array}\right] \leftarrow R_2 \leftrightarrow R_3$$

(i) Unique solution

$r(A) = 3 \quad r(A:b) = 3 \quad$ or $a \neq -3, \; b \neq 4$

$n = 3$

$(a+3)z = 0$ $\quad z = 0$

$(b-4)y = 1$ $\quad y = \dfrac{1}{b-4}$

$x + \dfrac{2}{b-4} = 2$

$x = 2 - \dfrac{2}{b-4}$

$x = 2\left(\dfrac{b-5}{b-4}\right)$

(ii) Infinite solutions

$r(A) = r(A:b) = 2$, $\quad a = -3, \quad b \neq 4$

$(b-4)y = 1$

$y = \dfrac{1}{b-4}$

Let $x = k$

$k + \dfrac{2}{b-4} + 3z = 2$

$3z = 2 - k - \dfrac{2}{b-4}$

$z = \dfrac{2}{3} - \dfrac{k}{3} - \dfrac{2}{3(b-4)}$

(iii) No solutions

$r(A) \neq r(A:b)$, $\quad b = 4 \quad$ and $\quad a = 3$ or $a \neq 3$

rank 2 $\quad$ rank 3

Q21. $x+y+z=1$
$x+y-2z=3$
$2x+y+z=2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 3 \\ 2 & 1 & 1 & 2 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -3 & 2 \end{array}\right]$$

$-3z=2$
$\boxed{z=-\frac{2}{3}}$

$-y-z=0$
$-y+\frac{2}{3}=0$
$\boxed{y=\frac{2}{3}}$

$x+\frac{2}{3}-\frac{2}{3}=1$
$\boxed{x=1}$

$(x,y,z)=(1,2/3,-2/3)$

Q22. $x+y+2z+3t=13$
$x-2y+z+t=8$
$3x+y+z-t=1$

$$[A:b] = \left[\begin{array}{cccc:c} 1 & 1 & 2 & 3 & 13 \\ 1 & -2 & 1 & 1 & 8 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-R_1} \left[\begin{array}{cccc:c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & -2 & -5 & -10 & -38 \end{array}\right]$$

$$\xleftarrow{R_3 - \frac{2}{3}R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & 0 & -\frac{13}{3} & -\frac{26}{3} & -\frac{104}{3} \end{array}\right]$$

$r(A) = 3 = r(A:b) < n = 4$

$\therefore$ consistent with ∞ solutions

Let $t = k$

$$-\frac{13}{\cancel{3}} z - \frac{26}{\cancel{3}} k = \frac{-104}{\cancel{3}}$$

$$-13z = 26k - 104$$

$$z = -2k + 8$$

$$-3y - 8 + \cancel{2k} - \cancel{2k} = -5$$

$$y = -1$$

$$x - 1 + 16 - 4k + 3k = 13$$

$$x + 15 - k = 13$$

$$x = k - 2$$

Q23. $x + z = 1$ what if RHS $= (1, 2, 0)$?

$x + y + z = 2$

$x - y + z = 1$

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$r(A) = 2$
$r(A:b) = 3$

$\therefore$ inconsistent

$x + z = 1$
$x + y + z = 2$
$x - y + z = 0$

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r(A) = 2 = r(A:b) < n = 3$

$\therefore$ consistent with ∞ solutions

let $z = k$ $\quad\quad y = 1$

$x + k = 1$
$x = 1 - k$

$(x, y, z) = (1-k, 1, k)$

Q24. Find the values of a and b

$$x + y + az = 2b$$
$$x + 3y + (2+2a)z = 7b$$
$$3x + y + (3+3a)z = 11b$$

i) what is trivial solution (all variables have to be 0)

ii) What is unique non-trivial solution (at least one nonzero solution)

iii) what is infinite set of solutions

iv) No solution

$$[A:b] = \begin{bmatrix} 1 & 1 & a & : & 2b \\ 1 & 3 & 2+2a & : & 7b \\ 3 & 1 & 3+3a & : & 11b \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & a & : & 2b \\ 0 & 2 & 2+a & : & 5b \\ 0 & -2 & 3 & : & 5b \end{bmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & a & : & 2b \\ 0 & 2 & 2+a & : & 5b \\ 0 & 0 & 5+a & : & 10b \end{bmatrix}$$

i) $b = 0$, $a \neq -5$

ii) $b \neq 0$, $a \neq -5$

iii) $b = 0$, $a = -5$

iv) $b \neq 0$, $a = -5$

Q25. Let $A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$ and $b = (b_1, b_2, b_3, b_4)$

(i) Find values of b such that $Ax=b$ is consistent

(ii) $b=(2,1,1,1)$ if $(x,0,0,1)$ is solution to $Ax+b$, what is x?

$$[A:b] = \left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & b_1 \\ -2 & 4 & 1 & 3 & b_2 \\ 0 & 0 & 1 & 1 & b_3 \\ 1 & -2 & 1 & 0 & b_4 \end{array}\right] \xrightarrow[R_4 - 1/3 R_1]{R_2 + 2/3 R_1} \left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & b_1 \\ 0 & 0 & 7/3 & 7/3 & b_2 + \frac{2}{3}b_1 \\ 0 & 0 & 1 & 1 & b_3 \\ 0 & 0 & 1/3 & 1/3 & b_4 - \frac{b_1}{3} \end{array}\right]$$

$$\xrightarrow[R_4 - 1/7 R_2]{R_3 - 3/7 R_2} \left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & b_1 \\ 0 & 0 & 7/3 & 7/3 & b_2 + 2/3\, b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3/7\, b_2 - 2/7\, b_1 \\ 0 & 0 & 0 & 0 & b_4 - b_1/3 - b_2/7 - \frac{2b_1}{21} \end{array}\right]$$

(i) $r(A) = r(A:b)$

$$b_3 - \frac{3}{7}b_2 - \frac{2}{7}b_1 = 0$$

$$\boxed{7b_3 - 3b_2 - 2b_1 = 0}$$

$$b_4 - \frac{b_1}{3} - \frac{b_2}{7} - \frac{2b_1}{21} = 0$$

$$21b_4 - 7b_1 - 3b_2 - 2b_1 = 0$$

$$\boxed{21b_4 - 9b_1 - 3b_2 = 0}$$

(ii)

$$\left[\begin{array}{cccc:l} 3 & -6 & 2 & -1 & b_1 \\ 0 & 0 & 7/3 & 7/3 & b_2 + 2/3\, b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3/7\, b_2 - 2/7\, b_1 \\ 0 & 0 & 0 & 0 & b_4 - b_1/3 - b_2/7 - \frac{2b_1}{21} \end{array}\right]$$

$b = (2,1,1,1)$

$$\left[\begin{array}{cccc:l} 3 & -6 & 2 & -1 & 2 \\ 0 & 0 & 7/3 & 7/3 & 1 + 4/3 \\ 0 & 0 & 0 & 0 & 1 - 3/7 - 4/7 \\ 0 & 0 & 0 & 0 & 1 - 2/3 - 1/7 - 4/21 \end{array}\right]$$

$$= \left[\begin{array}{cccc:l} 3 & -6 & 2 & -1 & 2 \\ 0 & 0 & 7/3 & 7/3 & 7/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$3x - 6\times 0 + 2\times 0 - 1 = 2$$

$$\boxed{x = 1}$$

$\boxed{\text{ELEMENTARY MATRICES}}$

Elementary matrix E_{ij} is obtained from I by performing a single elementary row operation

$$R_i - l_{ij} R_j \quad \text{where } l_{ij} \text{ is the multiplier}$$

$$\text{i.e. } I \rightarrow E_{ij}$$

eg: E_{32} : $R_3 - 2R_2$ multiplier

$i=3$ $j=2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}_{3\times 3}$$

at most one nonzero entry off the main diagonal

$$\boxed{E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U} \quad \text{from } A \text{ to } U$$

$$A = E_{21}^{-1} \cdot E_{31}^{-1} \, E_{32}^{-1} \, U \quad \text{from } U \text{ to } A$$

Q26. Write down the elementary matrices associated with the given system of equations

$$2u + v + 3w = -1$$
$$4u + v + 7w = 5$$
$$-6u - 2v - 12w = -2$$

Step 1: convert to matrix A

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}$$

Step 2: Convert to Upper Triangular Matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

equivalent to A

$$U = A \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \text{UTM}$$

Step 3: Identify multipliers

$$E_{21} = -2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for $R_2 = R_2 - 2R_1$

$$E_{31} = 3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$E_{32} = 1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Q27. Which elementary matrices put A into UTM U?

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 1/3 R_1]{R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix}$$

$$\xrightarrow{R_3 + 5/11 R_2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5/11 & 1 \end{bmatrix}$$

$$\boxed{E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U}$$

Q28. Which elementary matrices convert A to UTM U?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 + \frac{1}{2}R_1} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2/3\, R_2}$$

$$A \sim \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\downarrow R_4 + 3/4\, R_3$$

$$A \sim \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = U$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix}$$

$$\boxed{E_{43} \cdot E_{32} \cdot E_{21} \cdot A = U}$$

Q29. Which elementary matrices convert A to UTM U?

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 0 & 8 \\ -1 & -1 & 4 & -2 \\ -2 & -2 & 6 & -3 \end{bmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 + 2R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & 4 & 1 \\ 0 & -4 & 6 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & -4 & 6 & 3 \end{bmatrix} \xrightarrow{R_4 - 2R_2} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 + 2R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_3} E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

TRIANGULAR FACTORS

- To undo the steps of Gaussian Elimination and revert back to original matrix A from U
- Instead of subtracting elementary matrices, the inverses are subtracted from A
- $E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1} \ldots$ should be obtained by changing the signs of elementary matrices
- $$\underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{=L} U = A$$

triangular FACTORISATION - LU

- Any square matrix A can be factorised as $\boxed{A = LU}$ where

  asymmetric

  L: lower triangular matrix — 1's on diagonal

  U: upper triangular matrix — U's on diagonal

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

diagonal elements; asymmetric

- Introduced by Alan Turing
- One method: use multiplier coefficients from row transformations ($E_{21} \rightarrow l_{21}$, etc)
- $L\underbrace{Ux}_{z} = b \qquad Lz = b \qquad Ux = z$

Q30. Solve the following system of equations using LU decomposition method.

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\Big\downarrow R_3 - 3R_1 \mid R_2 - 4R_1$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} = U$$

To find L:

$$\left.\begin{aligned} E_{21} &\longrightarrow -4 \\ E_{31} &\longrightarrow -3 \\ E_{32} &\longrightarrow 2 \end{aligned}\right\} \xrightarrow{\text{change sign}} \begin{aligned} l_{21} &= 4 \\ l_{31} &= 3 \\ l_{32} &= -2 \end{aligned}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\boxed{LZ = b}$$

forward substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\boxed{z_1 = 1}$$

$$4 + z_2 = 6$$

$$\boxed{z_2 = 2}$$

$$3 - 4 + z_3 = 4$$

$$\boxed{z_3 = 5}$$

$Ux = z$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$-10x_3 = 5$$

$$\boxed{x_3 = -1/2}$$

$$-x_2 + 5/2 = 2$$

$$\boxed{x_2 = 1/2}$$

$$x_1 + 1/2 - 1/2 = 1$$

$$\boxed{x_1 = 1}$$

Q31. Solve the following systems of equations using LU decomposition method.

(i) $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$

(iii) $2u + v + 3w = -1$
$4u + v + 7w = 5$
$-6u - 2v - 12w = -2$

(i) $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$

$$Ax = b$$
$$LUx = b \qquad Ux = z$$
$$Lz = b$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 1/3 R_1]{R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix} \xrightarrow{R_3 + 5/11 R_2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$$

$$\boxed{Lz = b}$$

$\boxed{z_1 = 4}$ $8/3 + z_2 = 1$ $4/3 + 25/33 + z_3 = -8$

$\boxed{z_2 = -5/3}$ $\boxed{z_3 = -111/11}$

$$z = \begin{bmatrix} 4 \\ -5/3 \\ -111/11 \end{bmatrix}$$

$Ux = z$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5/3 \\ -111/11 \end{bmatrix}$$

$-\frac{8}{11} z = \frac{-111}{11}$ $\quad$ $-\frac{11}{\cancel{3}} y \quad -\frac{7}{\cancel{3}} \times \frac{111}{8} = \frac{-5}{\cancel{3}}$ $\quad$ $3x - \frac{67}{8} + \frac{222}{8} = 4$

$\boxed{z = \frac{111}{8}}$ $\quad$ $\boxed{y = -\frac{67}{8}}$ $\quad$ $\boxed{x = -\frac{41}{8}}$

(ii)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 + 2R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_2}$$

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$E_{21} \longrightarrow l_{21} = 4$

$E_{31} \longrightarrow l_{31} = -2$

$E_{32} \longrightarrow l_{32} = 2$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

$$Lz = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

$\boxed{z_1 = 2}$ $\quad 4 \times 2 + z_2 = 6$ $\quad -4 - 4 + z_3 = -1$

$\boxed{z_2 = -2}$ $\quad \boxed{z_3 = 7}$

$$Ux = z$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 7 \end{bmatrix}$$

$-2z = 7$ $\qquad$ $2y - 7/2 = -2$ $\qquad$ $x + 3/4 = 2$

$\boxed{z = -7/2}$ $\qquad$ $\boxed{y = 3/4}$ $\qquad$ $\boxed{x = 5/4}$

(iii) $\quad 2u + v + 3w = -1$

$4u + v + 7w = 5$

$-6u - 2v - 12w = -2$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\Big\downarrow R_3 + 3R_1 \quad R_2 - 2R_1$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

$$Lz = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$\boxed{z_1 = -1}$

$-2 + z_2 = 5$

$\boxed{z_2 = 7}$

$3 - 7 + z_3 = -2$

$\boxed{z_3 = 2}$

$Ux = z$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$-2z = 2$

$\boxed{w = -1}$

$-y - 1 = 7$

$\boxed{v = -8}$

$2x - 8 - 3 = -1$

$\boxed{u = 5}$

triangular FACTORISATION – LDU

symmetric

- Any square matrix A can be factorised as $\boxed{A = LDU}$ where
  - L: lower triangular matrix – 1's on diagonal
  - D: diagonal matrix – d's on diagonal
  - U: upper triangular matrix – 1's on diagonal

divide row by pivot

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Q32. Solve using LU factorisation

$$2x - 3y = 3$$
$$4x - 5y + z = 7$$
$$2x - y - 2z = 5$$

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\downarrow R_3 - 2R_2$$

$$E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U \qquad U_1 = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$L = [E_{32} \cdot E_{31} \cdot E_{21}]^{-1}$$

$$L = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \qquad U_1 = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$Lz = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$\boxed{z_1 = 3}$ $\qquad 6 + z_2 = 7 \qquad 3 + 2 + z_3 = 5$

$\boxed{z_2 = 1}$ $\qquad \boxed{z_3 = 0}$

$$U_1 x = z$$

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$-4z = 0 \qquad \boxed{y = 1} \qquad 2x - 3 = 3$

$\boxed{z = 0} \qquad \boxed{x = 3}$

Q33. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} \xrightarrow[\substack{R_3 + 2R_1 \\ R_4 + R_1}]{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_2,\ R_3 + 2R_2} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix} \xrightarrow{R_4 + 2R_3} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$A = LDU$$

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1/3 & -2/3 & 2/3 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q34. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 + 1/2 R_1} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2/3 R_2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_4 + 3/4 R_3}$$

$$U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$$

$$A = LU$$

$$\overset{A}{\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}} \cdot \overset{U}{\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}}$$

$$A = LDU$$

$$= \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}} \cdot \overset{D}{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Q35. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - R_1} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r \end{bmatrix}$$

$$R_4 - R_2 \downarrow R_3 - R_2$$

$$U = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

$$A = LDU$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-r & 0 & 0 \\ 0 & 0 & c-s & 0 \\ 0 & 0 & 0 & d-t \end{bmatrix} \begin{bmatrix} 1 & r/a & r/a & r/a \\ 0 & 1 & (s-r)/(b-r) & (s-r)/(b-r) \\ 0 & 0 & 1 & (t-s)/(c-s) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q36. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 + 1/3 R_1]{R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 7/3 & 5/3 \end{bmatrix}$$

$R_3 + \frac{7}{11} R_2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -7/11 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & 2/11 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -7/11 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & 2/11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -7/11 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -11/3 & 0 \\ 0 & 0 & 2/11 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 1 & 7/11 \\ 0 & 0 & 1 \end{bmatrix}$$

ROW EXCHANGES

- If zero appears in pivot position, row exchanges
- Row exchange is taken care of by Permutation Matrices P
- $A \neq LU$ but $PA = LU$ where P is a Permutation Matrix (identity matrix with rows in different order)
- Inverse of a permutation matrix = permutation matrix
- $P^{-1} = P^T$ $\qquad P_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ there are $2!$ PMs of order 2

Q37. Consider $y = b_1$, $2x + 3y = b_2$

$$Ax = b$$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

pivot $= 0$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Gaussian elimination fails; row exchange

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = Pb = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$$

$$PA = LU$$

$$U = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{cccc} P & A & = & L\,U \end{array}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A = P^{-1} L U = P^T L U$$

$$\boxed{\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}}$$

Q38. Factorise $PA = LU$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 0 & -1 \\ 0 & 5 & 7 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

elimination fails

PA

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$PA = LU$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}}$$

Q39. Factorise into LU and LDU

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ -2 & 5 & -4 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

GE fails

$$\downarrow R_2 \leftrightarrow R_3$$

$$U = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix} \xrightarrow[R_2+2R_1]{R_3-2R_1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$PA = LU$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}}_{PA} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

pivots already 1

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = LDU$$

$$= \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \overset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

inverses & transposes

- inverse B of a square matrix A is A^{-1}
- $AB = BA = I$ (identity matrix)

Properties

1. A^{-1} is unique for a matrix A
2. $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

   if $A = LU$, $\quad A^{-1} = U^{-1}L^{-1}$
3. A matrix A is invertible if and only if elimination produces n pivots with or without row exchanges (without permanent breakdown)

   $$Ax = b \Rightarrow x = A^{-1}b$$

   elimination solves $Ax=b$ without explicitly finding A^{-1}

Gauss-Jordan Method

- inverse of invertible matrix A is obtained by a set of row operations that transforms A to I and I to A^{-1}
- augmented matrix $[A : I]$
- convert A to U and reduce I to C.
- reduce U to I and reduce C to A^{-1}

$$\boxed{[A:I] \longrightarrow [U:C] \longrightarrow [I:A^{-1}]}$$

Q40. Compute A^{-1} using Gauss-Jordan Method

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -2 & 2 & 2 \end{bmatrix}$$

pivot

$$[A:I] = \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 & 1 & 0 \\ -2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 + R_1$ | $R_2 \rightarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 3 & 1 & 0 & 1 \end{array}\right]$$

pivot

$R_3 \rightarrow R_3 - 3R_2$

pivot

$$[U : C] = \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to R_1 + \tfrac{1}{3} R_3 \qquad R_2 \to R_2 + \tfrac{2}{3} R_3$$

pivot

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 0 & 10/3 & -1 & 1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 2/3 & 0 & -1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to \tfrac{1}{2} R_1 \qquad R_3 \to -\tfrac{1}{3} R_3$$

$$[I : A^{-1}] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/3 & 0 & -1/6 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & 1 & -7/3 & 1 & -1/3 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1/3 & 0 & -1/6 \\ 8/3 & -1 & 2/3 \\ -7/3 & 1 & -1/3 \end{bmatrix}}$$

Q41. Compute A^{-1} using Gauss-Jordan Elimination

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}$

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_1$ $\downarrow$ $R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \end{array}\right]$$

$\downarrow$ $R_3 \to R_3 + 3/5\, R_2$

$$[U:C] \quad \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & 0 & -3/5 & -1/5 & 3/5 & 1 \end{array}\right]$$

$R_1 \to R_1 + 5/3\, R_3$ $\downarrow$ $R_2 \to R_2 - 5/3\, R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 2/3 & 1 & 5/3 \\ 0 & -5 & 0 & -5/3 & 0 & -5/3 \\ 0 & 0 & -3/5 & -1/5 & 3/5 & 1 \end{array}\right]$$

$R_1 \to R_1 + 1/5\, R_2$

$$\downarrow$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/3 & 1 & 4/3 \\ 0 & -5 & 0 & -5/3 & 0 & -5/3 \\ 0 & 0 & -3/5 & -1/5 & 3/5 & 1 \end{array}\right]$$

$$R_3 \rightarrow -5/3\, R_3 \;\Big\downarrow\; R_2 \rightarrow -1/5\, R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/3 & 1 & 4/3 \\ 0 & 1 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 & -1 & -5/3 \end{array}\right]$$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

$$A = \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \rightarrow R_3 - R_1 \;\Big\downarrow\; R_2 \rightarrow R_2 - 2R_1$$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_1 \rightarrow R_1 - 3R_3 \;\Big\downarrow\; R_2 \rightarrow R_2 + 5R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 4 & 0 & -3 \\ 0 & -1 & 0 & -7 & 1 & 5 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\big|$$

$$\Big\downarrow R_1 \to R_1 + 2R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -10 & 2 & 7 \\ 0 & -1 & 0 & -7 & 1 & 5 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 \to -R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -10 & 2 & 7 \\ 0 & 1 & 0 & 7 & -1 & -5 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Transpose of a Matrix

- rows & columns interchange

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ -3 & 0 \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{bmatrix}_{2\times 3}$$

Properties

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $(A^{-1})^T = (A^T)^{-1}$
4. $(A \pm B)^T = A^T \pm B^T$
5. $(A^{-1})^T A^T = (AA^{-1})^T = I$

$\boxed{\text{Symmetric MATRIX}}$

- $A^T = A$
- if A is symmetric and A^{-1} exists, A^{-1} is also symmetric
- eg: $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Properties

- $(A^{-1})^T = A^{-1}$
- if A is symmetric and $A = LDU$, then

$$\boxed{A = A^T = LDL^T \quad (\because U = L^T \ \& \ L = U^T)}$$

Q42. Factorise into $A = LDU$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 7 \end{bmatrix} = U$$

$$\Big\downarrow R_3 \to \tfrac{1}{7}R_3 \,\big|\, R_2 \to -R_2$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}} \overset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix}} \overset{U = L^T}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}$$

Q43. For which 3 no.s 'c' is this matrix not invertible? ← non-singular

$P_1 \times P_2 \times P_3 = |A|$

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - c/2 R_1} \begin{bmatrix} 2 & c & c \\ 0 & c - c^2/2 & c - c^2/2 \\ 0 & 7-4c & -3c \end{bmatrix} \xrightarrow{R_3 - \frac{7-4c}{c-c^2/2} R_2} \begin{bmatrix} 2 & c & c \\ 0 & c - c^2/2 & c - c^2/2 \\ 0 & 0 & c-7 \end{bmatrix}$$

$$|A| = 2 \times \left(c - \frac{c^2}{2}\right) \times (c-7)$$

$$c - 7 \neq 0 \qquad c - \frac{c^2}{2} \neq 0$$

$$c\left(1 - \frac{c}{2}\right) \neq 0$$

$\boxed{c \neq 7}$ and $\boxed{c \neq 0}$ and $\boxed{c \neq 2}$

Q44. Use Gauss-Jordan Method to find A

$$A = \begin{bmatrix} 1 & a & b \\ 1 & a & 2 \\ 1 & 0 & b \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$[A^{-1} : I] = \left[\begin{array}{ccc:ccc} 1 & a & b & 1 & 0 & 0 \\ 1 & a & 2 & 0 & 1 & 0 \\ 1 & 0 & b & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 0 & 2-b & -1 & 1 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \end{array}\right]$$

$\downarrow R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & a & 0 & \frac{2}{2-b} & \frac{-b}{2-b} & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right] \xleftarrow{R_1 \to R_1 - \frac{b}{2-b} R_3} \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow \; R_1 \longrightarrow R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{b}{2-b} & \frac{-b}{2-b} & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right]$$

$$R_3 \to \frac{R_3}{2-b} \;\Big\downarrow\; R_2 \to -\frac{1}{a} R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{b}{2-b} & -\frac{b}{(2-b)} & 0 \\ 0 & 1 & 0 & \frac{1}{a} & 0 & -\frac{1}{a} \\ 0 & 0 & 1 & -\frac{1}{2-b} & \frac{1}{2-b} & 0 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\frac{b}{2-b} = 1 \quad \Rightarrow \quad b = 2-b$$

$$2b = 2$$

$$\boxed{b = 1}$$

$$\frac{1}{a} = 1 \quad \Rightarrow \quad \boxed{a = 1}$$